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Generalized KKM Theorem on H-Space with Applications*

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In this paper KKM theorem, Ky Fan's matching theorems, minimax inequalities, and coincidence theorems are further generalized, so as to unify and strengthen the corresponding results in recent works. © 1992 Academic Press, Inc.

1. INTRODUCTION AND PRELIMINARIES

The famous Knaster–Kuratowski–Mazurkiewicz Theorem [10] (KKM theorem, for short) was first generalized to the infinite dimensional cases by Ky Fan [6], and since then this theorem has become an importantly theoretical foundation of the KKM technique in dealing with nonlinear problems. Recently, Horvath [9] generalized the KKM theorem by replacing convexity assumptions with the merely topological property, i.e., contractibility. In [4] the author introduced the concept of generalized KKM mapping and obtained a generalization of F-KKM theorem in Hausdorff topological vector space.

The purpose of this paper is to establish a generalized KKM theorem in Horvath's abstract setting, so as to unify and strengthen the corresponding results mentioned above and to extend by Fan's matching theorems. As

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applications of these results we establish a general version of minimax inequalities and obtain some coincidence theorems with succinct summary.

DEFINITION 1 [1]. An H-space is a pair $(X, \{\Gamma_A\})$, where X is a topological space, and $\{\Gamma_A\}$ is a given family of nonempty contractible subsets of X, indexed by the finite subsets of X such that $A \subset A'$ implies $\Gamma_A \subset \Gamma_{A'}$.

Let $(X, \{\Gamma_A\})$ be an H-space. A subset $D \subset X$ is called H-convex (or weakly H-convex) relative to subset $C \subset X$, if for each finite subset $A \subset C$, it follows $\Gamma_A \subset D$ (or $\Gamma_A \cap D$ is nonempty and contractible). When $C = D$, then D is called H-convex (or weakly H-convex) briefly.

A subset $K \subset X$ is called H-compact, if for every finite subset $A \subset X$, there exists a compact weakly H-convex set $D \subset X$ such that $K \cup A \subset D$.

A subset $M \subset X$ is called compactly closed (compactly open), if M is closed (open) relative to every compact subset of X.

Remark 1. [1]. Hausdorff topological vector space, convex space [14], contractible space, and pseudo-convex space [8] are all the special cases of H-space.

LEMMA 1 [9]. *Let $(X, \{\Gamma_A\})$ be an H-space. Let $x_1, x_2, ..., x_n$ be any n points of X (not necessarily distinct). Then*

(1) *For a standard $(n-1)$-simplex $e_1 e_2 \cdots e_n$, there exists a continuous mapping $f: e_1 e_2 \cdots e_n \to X$, such that*

$$f(e_{i_1} e_{i_2} \cdots e_{i_k}) \subset \Gamma_{\{i_1, x_{i_2}, ..., x_{i_k}\}},$$

where $\{i_1, i_2, ..., i_k\}$ is any nonempty subset of $\{1, 2, ..., n\}$.

(2) *Let $M_1, M_2, ..., M_n$ be n compactly closed subsets of X. If for any $\{i_1, i_2, ..., i_k\} \subset \{1, 2, ..., n\}$, such that*

$$\Gamma_{\{x_{i_1}, x_{i_2}, ..., x_{i_k}\}} \subset \bigcup_{j=1}^{k} M_{i_j},$$

then $\bigcap_{i=1}^{n} M_i \neq \varnothing$.

LEMMA 2. *Let $(X, \{\Gamma_A\})$ be an H-space and let $M_1, M_2, ..., M_n$ be n compactly closed subsets of X such that*

$$\bigcup_{i=1}^{n} M_i = X.$$

Then for any n points $x_1, x_2, ..., x_n$ (not necessarily distinct) of X, there exists a subset $\{i_1, i_2, ..., i_k\} \subset \{1, 2, ..., n\}$ such that

$$\Gamma_{\{x_{i_1}, x_{i_2}, ..., x_{i_k}\}} \cap \bigcap_{j=1}^{k} M_{i_j} \neq \varnothing.$$

Proof. Consider an $(n-1)$-dimensional standard simplex $e_1 e_2 \cdots e_n$ of an Euclidian space R^n and the continuous mapping $f: e_1 e_2 \cdots e_n \to X$ as shown in Lemma 1. For any given $u \in e_1 e_2 \cdots e_n$, letting

$$I(u) = \{i: f(u) \in M_i\}, \qquad S(u) = \mathrm{co}\{e_i: i \in I(u)\},$$

it is obvious that $I(u) \neq \varnothing$, and so $S(u) \neq \varnothing$. Since $\bigcup_{i \notin I(u)} M_i$ is compactly closed,

$$U = e_1 e_2 \cdots e_n \backslash f^{-1}\left(\bigcup_{i \notin I(u)} M_i\right)$$

is an open neighborhood of u in $e_1 e_2 \cdots e_n$. If $u' \in U$, then $I(u') \subset I(u)$, and so $S(u') \subset S(u)$. Consequently, $S: e_1 e_2 \cdots e_n \to e_1 e_2 \cdots e_n$ is an upper semicontinuous mapping with nonempty compact convex values. Therefore by the Kakutani fixed point theorem there exists a point $u_0 \in e_1 e_2 \cdots e_n$ such that

$$u_0 \in S(u_0) = \mathrm{co}\{e_i: i \in I(u_0)\}.$$

If we take $x_0 = f(u_0)$, then $x_0 \in \Gamma_{\{x_i: i \in I(u_0)\}}$ and $x_0 \in M_i$, $\forall i \in I(u_0)$. Hence we have

$$\Gamma_{\{x_i: i \in I(u_0)\}} \cap \bigcap_{i \in I(u_0)} M_i \neq \varnothing.$$

This completes the proof.

2. Generalized KKM Theorem

Definition 2. Let X be a nonempty set, $(Y, \{\Gamma_A\})$ an H-space, and $F: X \to 2^Y$ a set-valued mapping. If for any finite set $\{x_1, ..., x_n\} \subset X$, correspondently, there exists a finite set $\{y_1, ..., y_n\} \subset Y$ such that for any subset $\{y_{i_1}, ..., y_{i_k}\} \subset \{y_1, ..., y_n\}$ $(1 \leqslant k \leqslant n)$,

$$\Gamma_{\{y_{i_1}, ..., y_{i_k}\}} \subset \bigcup_{j=1}^{k} F(x_{i_j}),$$

then F is called a generalized KKM mapping.

Remark. Definition 2 extends the same concept of generalized KKM mapping in [4] to the case of H-space, and it contains H-KKM mapping [1] as its special case.

The following theorem not only shows the basic properties of this kind of mapping but also gives some generalizations of F-KKM theorem [6], Kim [11, Theorem 2], and Horvath [9, Theorem 1 and Corollary 1].

THEOREM 1. *Let X be a nonempty subset, $(Y, \{\Gamma_A\})$ an H-space, and $F: X \to 2^Y$ a generalized KKM mapping satisfying one of the following conditions:*

(i) *For each $x \in X$, $F(x)$ is compactly closed in Y;*

(ii) *For each $x \in X$, $F(x)$ is compactly open in Y.*

Then the family of $\{F(x): x \in X\}$ of sets has the finite intersection property. Moreover, if we add the following condition to condition (i),

there exists an $x_0 \in X$ such that $F(x_0)$ is a compact set,

then $\bigcap_{x \in X} F(x) \neq \emptyset$.

Proof. (i) For any finite subset $\{x_1, ..., x_n\} \subset X$, since $F: X \to 2^Y$ is a generalized KKM mapping, there exists a finite subset $\{y_1, ..., y_n\} \subset Y$ such that for any subset $\{y_{i_1}, ..., y_{i_k}\} \subset \{y_1, ..., y_n\}$ $(1 \leqslant k \leqslant n)$ we have

$$\Gamma_{\{y_{i_1}, ..., y_{i_k}\}} \subset \bigcup_{j=1}^{k} F(x_{i_j}).$$

Then by Lemma 1, we have $\bigcap_{i=1}^{n} F(x_i) \neq \emptyset$. This means that $\{F(x): x \in X\}$ has the finite intersection property.

Moreover, if $F(x_0)$ is a compact subset, then $\{F(x) \cap F(x_0): x \in X\}$ is a family of compact sets with the finite intersection property. By the property of compact set it follows that

$$\bigcap_{x \in X} (F(x) \cap F(x_0)) \neq \emptyset, \qquad \text{i.e.,} \bigcap_{x \in X} F(x) \neq \emptyset.$$

(ii) If $\{F(x): x \in X\}$ has no finite intersection property, then there exists a finite subset $\{x_1, ..., x_n\} \subset X$ such that

$$\bigcap_{i=1}^{n} F(x_i) = \emptyset.$$

Let $G(x) = Y \backslash F(x)$, then $G(x_1), ..., G(x_n)$ are compactly closed subsets in Y, and

$$\bigcup_{i=1}^{n} G(x_i) = Y \Big\backslash \bigcap_{i=1}^{n} F(x_i) = Y.$$

By the generalized KKM property of F, for the $\{x_1, ..., x_n\}$ there exists $\{y_1, ..., y_n\} \subset Y$ such that for any subset $\{y_{i_1}, ..., y_{i_k}\} \subset \{y_1, ..., y_n\}$ $(1 \leqslant k \leqslant n)$ we have

$$\Gamma_{\{y_{i_1}, ..., y_{i_k}\}} \subset \bigcup_{j=1}^{k} F(x_{i_j}) = Y \Big\backslash \bigcap_{j=1}^{k} G(x_{i_j}).$$

By Lemma 2, for $\{y_1, ..., y_n\} \subset Y$, there exists $\{y_{i_1}, ..., y_{i_m}\} \subset \{y_1, ..., y_n\}$ such that

$$\Gamma_{\{y_{i_1}, ..., y_{i_m}\}} \cap \bigcap_{j=1}^{m} G(x_{i_j}) \neq \varnothing,$$

a contradiction. This completes the proof.

It should be pointed out that only under condition (i) or (ii) it can not ensure that the family $\{F(x): x \in X\}$ of sets has nonempty intersection. This can be seen from the following

EXAMPLE. Let $X = Y = R$ (the real line), $\Gamma_A = \text{co}\, A$, $F_1(x) = \{y \in R : y \geqslant |x|\}$, $F_2(x) = \{y \in R : y > |x|\}$. It is easy to verify that $F_1, F_2 : R \to 2^R$ both are generalized KKM mappings and F_1 is closed-valued and F_2 is open-valued, but $\bigcap_{x \in R} F_i(x) = \varnothing$ $(i = 1, 2)$.

In the sequel, let $\mathscr{C}(X, Y)$ be the set of all continuous functions from X into Y. The following theorems are all the useful consequences of Theorem 1.

THEOREM 2. *Let D be a nonempty subset of a compact H-space $(X, \{\Gamma_A\})$ and Y a topological space. Suppose that $G: D \to 2^Y$ satisfies the following conditions:*

(i) *For every $x \in D$, $G(x)$ is compactly open in Y;*

(ii) *$G(D) = Y$.*

Then for any $s \in \mathscr{C}(X, Y)$ there exists a nonempty finite set $\{x_1^0, ..., x_n^0\} \subset D$ such that for any $\{x_1, ..., x_n\} \subset X$ there exist some subset $\{x_{i_1}, ..., x_{i_k}\} \subset \{x_1, ..., x_n\}$ and an $x_0 \in \Gamma_{\{x_{i_1}, ..., x_{i_k}\}}$ satisfying the condition

$$s(x_0) \in \bigcap_{j=1}^{k} G(x_{i_j}^0).$$

Proof. Let $F(x) = s^{-1}(Y \backslash G(x))$. Suppose that the conclusion is false. Then for any finite set $\{x_1, ..., x_n\} \subset D$, there exists $\{\eta_1, ..., \eta_n\} \subset X$ such that for any subset $\{\eta_{i_1}, ..., \eta_{i_k}\} \subset \{\eta_1, ..., \eta_n\}$ $(1 \leqslant k \leqslant n)$ the following holds:

$$s(\Gamma_{\{\eta_{i_1}, ..., \eta_{i_k}\}}) \subset Y \Big\backslash \bigcap_{j=1}^{k} G(x_{i_j}),$$

i.e.,

$$\Gamma_{\{\eta_{i_1}, ..., \eta_{i_k}\}} \subset \bigcup_{j=1}^{k} F(x_{i_j}).$$

This implies that $F: D \to 2^X$ is a generalized KKM mapping with compact values. By Theorem 1 we have

$$\bigcap_{x \in D} F(x) = \bigcap_{x \in D} s^{-1}(Y \backslash G(x)) \neq \varnothing.$$

Therefore $\bigcap_{x \in D} (Y \backslash G(x)) = Y \backslash G(D) \neq \varnothing$. This contradicts condition (ii).
This completes the proof.

COROLLARY 1. *Let D be a nonempty subset of an H-space $(X, \{\Gamma_A\})$, Y a topological space and $G: D \to 2^Y$ satisfies the following conditions*:

(i) *For every $x \in D$, $G(x)$ is compactly open in Y*;

(ii) $G(D) = Y$;

(iii) *There exist an H-compact set $L \subset X$ and a compact set $K \subset Y$ such that $Y \backslash G(L \cap D) \subset K$.*

Then for any $s \in \mathscr{C}(X, Y)$ there exist a nonempty finite subset $\{x_1, ..., x_n\} \subset D$ and an $x_0 \in \Gamma_{\{x_1, ..., x_n\}}$ such that $s(x_0) \in \bigcap_{i=1}^{n} G(x_i)$.

Proof. Since $Y \backslash G(L \cap D)$ is a closed subset of the compact set K,

$$Y \backslash G(L \cap D) = G(D) \backslash G(L \cap D) = \bigcup_{x \in D \backslash L} G(x).$$

Hence there exists a finite subset $\tilde{A} \subset D \backslash L$ such that

$$Y \backslash G(L \cap D) \subset \bigcup_{x \in \tilde{A}} G(x).$$

Let $D_1 = (L \cap D) \cup \tilde{A}$, then $G(D_1) = Y$. It follows from the H-compactness of L that there exists a compact weakly H-convex set $X_1 \supset D_1$. Hence for the compact H-space $(X_1, \{\Gamma_{A \cap X_1} \cap X_1\})$ and the mapping $G: D_1 \to 2^Y$ using Theorem 2, the conclusion of Corollary 1 can be obtained immediately.

Remark 2. Theorem 2 and Corollary 1 extend the matching theorem for open covering of [15] to the case of H-space.

THEOREM 3. *Let* $(X, \{\Gamma_A\})$ *be an* H*-space*, D *a nonempty subset of* X, *and* Y *a topological space. Suppose that* $F: D \to 2^Y$ *and* $s \in \mathscr{C}(X, Y)$ *satisfy the following conditions*:

(i) *for each weakly* H*-convex subset* $X' \subset X$, *if* $D \cap X' \neq \varnothing$, *then the mapping defined by*

$$\tilde{F}: D \cap X' \to 2^{X'}, \qquad x \mapsto s^{-1}F(x) \cap X'$$

is a generalized KKM mapping on H*-space* $(X', \{\Gamma_{A \cap X'} \cap X'\})$;

(ii) *for every* $x \in D$, $F(x)$ *is compactly closed in* Y;

(iii) *there exists an* H*-compact set* $L \subset X$ *and a compact set* $K \subset Y$ *such that for every weakly* H*-convex set* X_0 *with* $L \subset X_0 \subset X$ *the following holds*:

$$\bigcap_{x \in D \cap X_0} (F(x) \cap s(X_0)) \subset K.$$

Then $\bigcap_{x \in D} F(x) \neq \varnothing$.

Proof. It is sufficient to prove that $\bigcap_{x \in D} (F(x) \cap K) \neq \varnothing$. By condition (ii), $F(x) \cap K$ is closed in the compact set K. Hence it suffices to prove that for any finite set $A \subset D$, $\bigcap_{x \in A} (F(x) \cap K) \neq \varnothing$.

Let $A \subset D$ be any given finite set and $X_0 \supset L \cup A$ a compact weakly H-convex set. By condition (iii) we have

$$\bigcap_{x \in D \cap X_0} (F(x) \cap s(X_0)) \subset K.$$

Hence we have

$$\bigcap_{x \in A} (F(x) \cap K) \supset \bigcap_{x \in D \cap X_0} (F(x) \cap K) \supset \bigcap_{x \in D \cap X_0} (F(x) \cap s(X_0))$$

$$\supset s\left(\bigcap_{x \in D \cap X_0} (s^{-1}F(x) \cap X_0)\right) = s\left(\bigcap_{x \in D \cap X_0} \tilde{F}(x)\right).$$

Therefore in order to prove the desired conclusion, it suffices to prove that $\bigcap_{x \in D \cap X_0} \tilde{F}(x) \neq \varnothing$.

By condition (i), $\tilde{F}: D \cap X_0 \to 2^{X_0}$ is a generalized KKM mapping. It follows from condition (ii) and the compactness of $s(X_0)$ that for any $x \in D \cap X_0$, $\tilde{F}(x)$ is compact. By virtue of Theorem 1, we have $\bigcap_{x \in D \cap X_0} \tilde{F}(x) \neq \varnothing$. This completes the proof.

Remark 3. We would like to point out that if $F: D \to 2^Y$ is an H-KKM mapping, then it must satisfy the condition (i) of Theorem 3. (In fact, $\tilde{F}: D \cap X' \to 2^{X'}$ is also an H-KKM mapping on H-space $(X', \{\Gamma_{A \cap X'} \cap X'\})$.) Hence Theorem 3 generalizes [1, Theorem 1] and Park [15, Theorem 3] and it even more contains Lassonde [14, Theorem I] and Fan [7, Theorem 4] as its special cases.

Theorem 3 can be restated as follows:

COROLLARY 2. *Let D be a nonempty subset of an H-space $(X, \{\Gamma_A\})$, Y a topological space, $M \subset X \times Y$, $s \in \mathscr{C}(X, Y)$. If the following conditions are satisfied*:

(i) *for each weakly H-convex set $X' \subset X$, if $D \cap X' \neq \varnothing$, then for any finite set $\{x_1, ..., x_n\} \subset D \cap X'$, there exists $\{y_1, ..., y_n\} \subset X'$ such that for any subset $\{y_{i_1}, ..., y_{i_k}\} \subset \{y_1, ..., y_n\}$ and $x \in \Gamma_{\{y_{i_1}, ..., y_{i_k}\}} \cap X'$,*

$$(x_{i_j}, s(x)) \in M \qquad \text{for some } j \ (1 \leqslant j \leqslant k);$$

(ii) *for each $x \in D$, the set $\{y \in Y : (x, y) \in M\}$ is compactly closed in Y.*

Then for any H-compact set $L \subset X$, either there exists an $y_1 \in Y$ such that

$$(x, y_1) \in M, \qquad \text{for each} \quad x \in D;$$

or for any compact set $K \subset Y$, there exist a weakly H-convex set $X_0 \supset L$ and an $x_0 \in X_0$ such that $y_2 = s(x_0) \in Y \backslash K$ and

$$(x, y_2) \in M \qquad \text{for each} \quad x \in D \cap X_0.$$

Proof. Letting $F(x) = \{y \in Y : (x, y) \in M\}$, then $F: D \to 2^Y$ satisfies conditions (i), and (ii) of Theorem 2, therefore either the conclusion of Theorem 2 holds or the condition (iii) of Theorem 2 does not hold. This completes the proof.

By virtue of Theorem 1(ii) we are now in a position to give the following theorem whose corollary generalizes the matching theorem for closed covering of [15].

THEOREM 4. *Let D be a nonempty subset of a H-space $(X, \{\Gamma_A\})$, Y a topological space, and $F: D \to 2^Y$ and $s \in \mathscr{C}(X, Y)$ satisfy the following conditions*:

(i) *for each $x \in D$, $F(x)$ is compactly open in Y;*

(ii) *$\tilde{F}: x \in D \mapsto s^{-1}F(x)$ is a generalized KKM mapping.*

Then the family $\{F(x): x \in D\}$ of sets has the finite intersection property.

Proof. Since $\tilde{F}: D \to 2^X$ is a generalized KKM mapping with compactly open-valued, by Theorem 1, for any finite subset $A \subset D$ we have

$$\varnothing \neq \bigcap_{x \in A} \tilde{F}(x) = s^{-1}\left(\bigcap_{x \in A} F(x)\right),$$

and so $\bigcap_{x \in A} F(x) \neq \varnothing$. This completes the proof.

COROLLARY 3. *Let $(X, \{\Gamma_A\})$ be an H-space, Y a topological space, and $s \in \mathscr{C}(X, Y)$. Let $C_1, \ldots, C_n$ be n compactly closed subsets of Y satisfying*

$$\bigcup_{i=1}^{n} C_i = Y.$$

Then for any n points $x_1, \ldots, x_n$ of X (not necessarily distinct), there exists a subset $\{x_{i_1}, \ldots, x_{i_k}\} \subset \{x_1, \ldots, x_n\}$ such that

$$s(\Gamma_{\{x_{i_1}, \ldots, x_{i_k}\}}) \cap \bigcap_{j=1}^{k} C_{i_j} \neq \varnothing.$$

Proof. Suppose that the conclusion is false. Then there exists $\{x_1, \ldots, x_n\} \subset X$ such that for any nonempty subset $\{x_{i_1}, \ldots, x_{i_k}\} \subset \{x_1, \ldots, x_n\}$ we have

$$s(\Gamma_{\{x_{i_1}, \ldots, x_{i_k}\}}) \cap \bigcap_{j=1}^{k} C_{i_j} = \varnothing.$$

Consequently,

$$s(\Gamma_{\{x_{i_1}, \ldots, x_{i_k}\}}) \subset Y \Big\backslash \bigcap_{j=1}^{k} C_{i_j} = \bigcup_{j=1}^{k} (Y \backslash C_{i_j}).$$

Letting $D = \{x_1, \ldots, x_n\}$ and $F(x_i) = Y \backslash C_i$, then $F: D \to 2^Y$ satisfies all conditions of Theorem 4. Hence we have

$$\varnothing \neq \bigcap_{i=1}^{n} F(x_i) = Y \Big\backslash \bigcup_{i=1}^{n} C_i.$$

This contradicts $\bigcup_{i=1}^{n} C_i = Y$. The proof is complete.

3. MINIMAX INEQUALITIES AND COINCIDENCE THEOREMS

As applications, in this section we shall use the results presented in Section 2 to establish an abstract version of Ky Fan's minimax inequality.

THEOREM 5. *Let $(X, \{\Gamma_A\})$ be an H-space, Y a topological space, (E, C) a topological Riesz space, where C is a closed cone with $\mathring{C} \neq \varnothing$ ($\mathring{C}$ denotes the interior of C). Let $\alpha, \beta \in E$ be two given points and $f, g: X \times X \to E$ satisfy the following conditions*:

(i) *For every $y \in Y$, the set $\{x \in X : f(x, y) \in \alpha + \mathring{C}\}$ is H-convex relative to the set $\{x \in X : g(x, y) \in \beta + \mathring{C}\}$*;

(ii) *For each $x \in X$, the set $\{y \in Y : g(x, y) \in \beta + \mathring{C}\}$ is compactly open*;

(iii) *There exist an H-compact set $L \subset X$ and a compact set $K \subset Y$ such that*

$$M = \{y \in Y : g(x, y) \notin \beta + \mathring{C}, \forall x \in L\} \subset K.$$

Then one of the following conclusions holds:

(1) *There exists an $\bar{y} \in M$ such that*

$$g(x, \bar{y}) \notin \beta + \mathring{C}, \qquad \forall x \in X;$$

(2) *For any $s \in \mathscr{C}(X, Y)$ there exists an $\bar{x} \in X$ such that*

$$f(\bar{x}, s(\bar{x})) \in \alpha + \mathring{C}.$$

Proof. For $x \in X$ let

$$U(x) = \{y \in Y : g(x, y) \in \beta + \mathring{C}\},$$
$$V(x) = \{y \in Y : f(x, y) \in \alpha + \mathring{C}\}.$$

Suppose that the conclusion (1) is false. Then for any $y \in M$, there exists an $x \in X$ such that

$$g(x, y) \in \beta + \mathring{C}, \qquad \text{i.e., } y \in U(x),$$

and so $M \subset \bigcup_{x \in X} U(x)$. However, since

$$M = \bigcap_{x \in L} \{y \in Y : g(x, y) \notin \beta + \mathring{C}\} = \bigcap_{x \in L} (Y \backslash U(x)) = Y \Big\backslash \bigcup_{x \in L} U(x),$$

this implies that $Y \backslash \bigcup_{x \in L} U(x) \subset \bigcup_{x \in X} U(x)$. Hence we have $\bigcup_{x \in X} U(x) = Y$. Consequently

(1°) by condition (ii), for each $x \in X$, $U(x)$ is compactly open in Y;

(2°) $U(X) = Y$;

(3°) by condition (iii), there exist an H-compact set $L \subset X$ and a compact set $K \subset Y$ such that

$$Y \backslash U(L) = M \subset K.$$

By virtue of Corollary 1, for any $s \in \mathscr{C}(X, Y)$ there exist a nonempty finite subset $\{x_1, ..., x_n\} \subset X$ and an $\bar{x} \in \Gamma_{\{x_1, ..., x_n\}}$ such that

$$s(\bar{x}) \in \bigcap_{i=1}^{n} U(x_i).$$

Hence we have

$$x_i \in U^{-1}(s(\bar{x})), \qquad i = 1, ..., n,$$

i.e.,

$$\{x_1, ..., x_n\} \subset U^{-1}(s(\bar{x})).$$

Moreover by condition (i) we have

$$\bar{x} \in \Gamma_{\{x_1, ..., x_n\}} \subset V^{-1}(s(\bar{x})).$$

Hence $s(\bar{x}) \in V(\bar{x})$, i.e., $f(\bar{x}, s(\bar{x})) \in \alpha + \mathring{C}$. Therefore the conclusion (2) is true.

This completes the proof.

THEOREM 6. *Let $(X, \{\Gamma_A\})$ be an H-space, Y a topological space, E a Riesz space α, $\beta \in E$ two given elements, and f, g: $X \times Y \to E$. If the following conditions are satisfied:*

(i) *for any $y \in Y$, the set $\{x \in X : f(x, y) \nleqslant \alpha\}$ is H-convex relative the set $\{x \in X : g(x, y) \nleqslant \beta\}$;*

(ii) *for each $x \in X$, the set $\{y \in Y : g(x, y) \leqslant \beta\}$ is compactly closed;*

(iii) *there exist an H-compact set $L \subset X$ and a compact set $K \subset Y$ such that*

$$M = \{y \in Y : g(x, y) \leqslant \beta, \forall x \in L\} \subset K,$$

then one of the following conclusions holds:

(1) *there exists an $\bar{y} \in M$ such that*

$$g(x, \bar{y}) \leqslant \beta, \qquad \forall x \in X;$$

(2) *for any $s \in \mathscr{C}(X, Y)$, there exists an $\bar{x} \in X$ such that*

$$f(\bar{x}, s(\bar{x})) \nleqslant \alpha.$$

In addition, if the following condition is satisfied:

(iv) *E is order complete and there exists some $s \in \mathscr{C}(X, Y)$ such that*

$$\alpha = \beta = \sup_{x \in X} f(x, s(x)),$$

if the infimum of the mapping $y \mapsto \sup_{x \in X} g(x, y)$ *exists on* M, *then we have*

$$\inf_{y \in M} \sup_{x \in X} g(x, y) \leqslant \sup_{x \in X} f(x, s(x)).$$

Proof. For any $x \in X$ let

$$U(x) = \{y \in Y : g(x, y) \nleqslant \beta\};$$
$$V(x) = \{y \in Y : f(x, y) \nleqslant \alpha\}.$$

Suppose that the conclusion (1) is false. Then imitating the proof of Theorem 5 we can prove that the conclusion (2) holds.

Moreover, if condition (iv) is satisfied, then it is obvious that the conclusion (2) does not hold. Therefore there exists an $\bar{y} \in M$ such that

$$g(x, \bar{y}) \leqslant \beta = \sup_{x \in X} f(x, s(x)), \qquad \forall x \in X.$$

Since E is order complete, it follows from the preceding inequality that $\sup_{x \in X} g(x, \bar{y})$ exists and

$$\sup_{x \in X} g(x, \bar{y}) \leqslant \sup_{x \in X} f(x, s(x)).$$

If $\inf_{y \in M} \sup_{x \in X} g(x, y)$ exists, then it is obvious that it $\leqslant \sup_{x \in X} f(x, s(x))$.

This completes the proof.

Remark 4. Theorem 5 and Theorem 6 contain [1, Theorem 3], [1, Theorem 4 and Corollary 1], respectively; Yen [19, Theorem 1] and Shih and Tan [17, Theorem 3] are even more special cases of Theorem 6. Moreover, in Bardaro and Ceppitelli [2, Theorem 2] if we take

$$f_1(x, y) = g_1(x, y) = f(x, y) - \varphi(x) + \varphi(y): X \times X \to E$$

and

$$\alpha = \beta = 0, s = I_X,$$

then by [2, Propositions 1, 7, 8] it is easy to know that under the conditions of [2, Theorem 2], f_1, g_1 satisfy all conditions of Theorem 6. This shows that Theorem 6 is also a generalization of [2, Theorem 2].

In the sequel, we shall give some coincidence theorems.

THEOREM 7. *Let* D *be a nonempty subset of an* H-*space* $(X, \{\Gamma_A\})$, Y *a topological space. Let* $s \in \mathscr{C}(X, Y)$ *and* $U, V: D \to 2^Y$ *satisfy the following conditions*:

(i) *For each* $x \in D$, $U(x)$ *is compactly open in* Y;

(ii) *For each* $x \in X$, $V^{-1}(s(x))$ *is H-convex relative to* $U^{-1}(s(x))$;

(iii) *There exist an H-compact set* $L \subset X$ *and a compact set* $K \subset Y$ *such that*

$$U^{-1}(y) \neq \emptyset, \qquad \forall y \in \overline{s(X) \cap K},$$

and for each finite set $A \subset D$, *when* $x \in L_A \setminus s^{-1}(K)$ (*where* L_A *is a compact weakly H-convex set with* $L \cup A \subset L_A$) *we have*

$$U^{-1}(s(x)) \cap L_A \cap D \neq \emptyset.$$

Then there exists an $\bar{x} \in D$ *such that* $s(\bar{x}) \in V(\bar{x})$.

Proof. Since for each $y \in \overline{s(X) \cap K}$, $U^{-1}(y) \neq \emptyset$, this implies that $\overline{s(X) \cap K} \subset \bigcup_{x \in D} U(x)$. Since $\overline{s(X) \cap K}$ is compact, there exists a finite set $A_1 \subset D$ such that

$$\overline{s(X) \cap K} \subset \bigcup_{x \in A_i} U(x).$$

On the other hand, since L is an H-compact set, there exists a weakly H-convex set L_{A_1} such that $L_{A_1} \supset L \cup A_1$.

Now we prove that L_{A_1} has the property

$$L_{A_1} \subset s^{-1}U(L_{A_1} \cap D).$$

In fact, for any $x \in L_{A_1}$, if $x \in s^{-1}(K)$, and so $s(x) \in K$. Hence we have

$$s(x) \in U(A_1) \subset U(L_{A_1} \cap D), \qquad \text{i.e.,} \quad x \in s^{-1}U(L_{A_1} \cap D).$$

If $x \notin s^{-1}(K)$, by condition (iii), we have $U^{-1}(s(x)) \cap L_{A_1} \cap D \neq \emptyset$, and so

$$s(x) \in U(L_{A_1} \cap D), \qquad \text{i.e.,} \quad x \in s^{-1}U(L_{A_1} \cap D).$$

Summing up the proof as stated above we have $L_{A_1} \subset s^{-1}U(L_{A_1} \cap D)$.

Letting $\tilde{U}(x) = s^{-1}U(x) \cap L_{A_1}$, we know that the mapping $\tilde{U}: L_{A_1} \cap D \to 2^{L_{A_1}}$ on compact H-space $(L_{A_1}, \{\Gamma_{A \cap L_{A_1}} \cap L_{A_1}\})$ satisfies the following conditions:

(i) $\tilde{U}(x)$ is an open set;

(ii) $\tilde{U}(L_{A_1} \cap D) = L_{A_1}$.

By Theorem 2, there exist a nonempty finite subset $\{x_1, ..., x_n\} \subset L_{A_1} \cap D$ and an $\bar{x} \in \Gamma_{\{x_1, ..., x_n\}} \cap L_{A_1}$ such that

$$\bar{x} \in \bigcap_{i=1}^{n} \tilde{U}(x_i) \subset \bigcap_{i=1}^{n} s^{-1}U(x_i).$$

This means that

$$s(\bar{x}) \in U(x_i) \qquad \text{or} \qquad x_i \in U^{-1}(s(\bar{x})),\ i = 1, 2, ..., n.$$

By condition (ii) we have

$$\bar{x} \in \Gamma_{\{x_1, ..., x_n\}} \subset V^{-1}(s(\bar{x})),$$

and so $s(\bar{x}) \in V(\bar{x})$. This completes the proof.

COROLLARY 4. *Let D be a nonempty subset of an H-space $(X, \{\Gamma_A\})$, Y a topological space, and $U, V: D \to 2^Y$ satisfy the following conditions:*

(i) *For each $x \in D$, $U(x)$ is compactly open in Y;*

(ii) *For each $y \in Y$, $V^{-1}(y)$ is H-convex relative to $U^{-1}(y)$;*

(iii) $U(D) = Y$;

(iv) *There exist an H-compact set $L \subset X$ and a compact $K \subset Y$ such that for any finite subset $A \subset D$*

$$Y \backslash K \subset U(L_A \cap D),$$

where L_A is a compact weakly H-convex set containing $L \cup A$.

Then for each $s \in \mathscr{C}(X, Y)$, there exists an $\bar{x} \in D$ such that $s(\bar{x}) \in V(\bar{x})$.

Proof. If $x \in L_A \backslash s^{-1}(K)$ then $s(x) \notin K$. By condition (iv), $s(x) \in U(L_A \cap D)$, and so $U^{-1}(s(x)) \cap L_A \cap D \neq \emptyset$. It is obvious that the rest of the conditions of Theorem 7 are satisfied. This completes the proof.

Remark 5. As we have noted, Corollary 4 contains the generalized Fan–Browder fixed point theorem, i.e., [15, Theorem 6] as a special case. Therefore Theorem 7 unifies and strengthens Takahashi [18, Theorem 2.5], Lassonde [14, Theorem 1.1], Ben-El-Mechaiekh, Deguire, and Granas [3, Theorem 1], Simons [16, Theorem 4.3], Ko and Tan [13, Theorem 3.1], Park [15, Theorem 6], and others.

THEOREM 8. *Let D be a nonempty subset of an H-space $(X, \{\Gamma_A\})$, Y a topological space, and $s \in \mathscr{C}(X, Y)$, $U, V: D \to 2^Y$ satisfy the following conditions:*

(i) *for each $x \in D$, $U(x)$ is compactly closed in Y;*

(ii) *for each $x \in X$, $V^{-1}(s(x))$ is H-convex relative to $U^{-1}(s(x))$;*

(iii) *there exists a finite set $A_1 \subset D$ such that $\bigcup_{x \in A_1} U(x) = Y$.*

Then there exists an $\bar{x} \in D$ such that $s(\bar{x}) \in V(\bar{x})$.

Proof. By conditions (i), (iii), and Corollary 3, there exists a subset $A \subset A_1$ such that

$$s(\Gamma_A) \cap \bigcap_{x \in A} U(x) \neq \varnothing.$$

Hence there exists an $\bar{x} \in \Gamma_A$ such that $A \subset U^{-1}(s(\bar{x}))$. By condition (ii), $\bar{x} \in \Gamma_A \subset V^{-1}(s(\bar{x}))$, and so we have $s(\bar{x}) \in V(\bar{x})$. This completes the proof.

COROLLARY 5. *Let $(X, \{\Gamma_A\})$ be an H-space, Y a topological space. Let $s \in \mathscr{C}(X, Y)$ and P, Q: $Y \to 2^Y$ satisfy the following conditions*:

(i) *for each $x \in X$, $P^{-1}(x)$ is compactly closed in Y*;

(ii) *for each $x \in X$, $Q(s(x))$ is H-convex relative to $P(s(x))$*;

(iii) *there exists a finite subset $A_1 \subset X$ such that $P^{-1}(A_1) = Y$.*

Then there exists an $\bar{x} \in X$ such that $\bar{x} \in Q(s(\bar{x}))$.

Proof. Let $U(x) = P^{-1}(x)$, and $V(x) = Q^{-1}(x)$. Then U and V satisfy all conditions of Theorem 8. Then the conclusion of Corollary 5 follows from Theorem 8 immediately.

Theorem 8 and Corollary 5 extend [15, Theorem 9] and its contrapositive form to *H*-space. Especially, Theorem 8 contains Kim [11, Theorem 3; 12, Theorem 4] as its special cases.

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